

Possible Origin of Lognormal Distributions in Gamma-Ray Bursts

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ABSTRACT

We show that if the intrinsic break energy of Gamma-Ray Bursts (GRBs) is determined by the product of more than three random variables the observed break energy distribution becomes almost lognormal including the redshift effect because of the central limit theorem. The excess from the lognormal distribution at the low break energy is possibly due to the high redshift GRBs. The same argument may also apply to the pulse duration, the interval between pulses and so on.

Subject headings: gamma rays: bursts — gamma rays: theory

1. INTRODUCTION

Among the statistical properties of the observed quantities, lognormal distributions are frequently seen in Gamma-Ray Bursts (GRBs). The lognormal distribution may be defined as the distribution of a random variable x whose logarithm is normally distributed,

$$f(x)dx = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right] d\log x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases} \quad (1)$$

where $f(x)$ is the probability density function for x , and μ and σ^2 are the sample mean and the variance of $\log x$ (e.g., Crow & Shimizu 1988). This distribution is unimodal and positively skew. McBreen, Hurley, Long & Metcalfe (1994) pointed out that the total duration of the long and short bursts and the time interval between pulses are consistent with the lognormal distributions. Li & Fenimore (1996) showed that the pulse fluence and the pulse interval distributions within each burst are consistent with lognormal distributions. Nakar & Piran (2001) found that the pulse duration also have a lognormal distribution. The break energy distribution is also lognormal (Preece et al. 2000; see below).

Lloyd, Fryer & Ramirez-Ruiz (2001) suggested that $\sim 10\%$ of GRBs might have the redshift larger than 6 so that the redshift distribution might be wide. Therefore it is quite strange that the observed break energy distribution and the duration distribution are lognormal since it does not seem that the observed lognormal distribution reflects the redshift distribution of the GRBs. Even if the break energy distribution is lognormal at the source, the observed break energy should be smaller than the intrinsic one by a factor of $(1+z)$ while the observed duration should be longer than the intrinsic one by a factor of $(1+z)$. These factors change by order of unity between $z=0$ and $z\sim 6$.

In this Letter, we will consider a possible origin of the observed lognormal distributions in GRBs from the viewpoint of the central limit theorem.¹

2. BREAK ENERGY DISTRIBUTIONS

Figure 1 shows the histogram of the break energy E_b taken from the electronic edition of Preece et al. (2000).² The χ^2 test of all data gives the probability of 1.4×10^{-185} (the reduced χ^2 is 16.5 with 66 degrees of freedom) that the data was taken from the lognormal distribution. Therefore the null hypothesis that the break energy distribution is lognormal fails. However, if we exclude the data in the high and low energy ends, the fit becomes good as shown in Figure 1. The χ^2 test of the data between 70.8 keV and 708 keV gives the probability of 0.497 (the reduced χ^2 is 0.963 with 17 degrees of freedom) that the data was taken from the lognormal distribution, with $\mu = 2.38 \pm 0.004$ ($E_b \simeq 238$ keV) and $\sigma = 0.240 \pm 0.004$ (1σ width is between 137 keV and 413 keV). The improvement of the lognormal fit to the break energy distribution excluding the high and low energy ends may suggest that the soft and hard bursts originate from a different class of GRBs or emission mechanisms. Anyway, hereafter we will assume that the observed break energy distribution is lognormal.

Now let us assume that the intrinsic break energy distribution is lognormal. Next we numerically calculate the observed break energy distribution assuming that the redshift

¹The lognormal distribution of the time interval between pulses and the pulse fluence might be reproduced by a fine tuning of model parameters (Spada, Panaitescu & Mészáros 2000).

²It is not known whether the paucity of the soft and hard bursts is real or not, because harder bursts have fewer photons (Cohen, Piran & Narayan 1998; Lloyd & Petrosian 1999; but see Brainerd et al. 1999) and there may exist relatively many soft bursts with low luminosities, so called X-ray rich GRBs (or X-ray flashes or Fast X-ray transients) (Strohmayer et al. 1998; Heise et al. 2001; Kippen et al. 2001). Here we assume that the selection effect is small.

distribution has the form,

$$f(z)dz = \begin{cases} A(1+z)^{a-1}(1+z_0)^{b-1}dz, & \text{if } 0 < z < z_0, \\ A(1+z_0)^{a-1}(1+z)^{b-1}dz, & \text{if } z_0 < z, \end{cases} \quad (2)$$

where $b < 0$. This redshift distribution rises as $\propto (1+z)^{a-1}$ to a redshift of z_0 and then declines as $\propto (1+z)^{b-1}$, and it is similar to that in Figure 8 of Lloyd, Fryer & Ramirez-Ruiz (2001). To mimic Figure 1, we generate 155 bursts with each burst having 35 spectra for each realization. We take the mean and the variance of the intrinsic break energy distribution so that the mean and the variance of the observed break energy are close to that in Figure 1. For each realization, we make a χ^2 fitting to obtain the probability that the data is taken from the lognormal distribution. As in Figure 1, we use the data between 70.8 keV and 708 keV for the χ^2 test. From this simulation, we have found that the average probability can reach ~ 0.5 even when the variance of the redshift distribution is comparable to the observed one $\sigma_z^2 \sim (0.240)^2$, although it is slightly stronger condition to preserve the lognormal form than $\sigma_z^2 < (0.240)^2$. At first glance it is strange that the simulations do not reflect the redshift distribution contrary to the argument in Section 1. In the next section we will show that it is not strange but natural due to the central limit theorem.

Figure 2 shows the histogram of the observed break energy for one experimental realization with $(z_0, a, b) = (3, 3, -3)$. The χ^2 test of the data between 70.8 keV and 708 keV gives the probability of 0.128 (the reduced χ^2 is 1.39 with 17 degrees of freedom) that the data was taken from the lognormal distribution, with $\mu = 2.37 \pm 0.004$ ($E_b \simeq 235$ keV) and $\sigma = 0.256 \pm 0.004$ (1σ width is between 130 keV and 423 keV). It is interesting to note the excess of soft bursts relative to the lognormal fit as in Figure 1. The average redshifts of these soft bursts are relatively high.

3. LOGNORMAL DISTRIBUTIONS

The standard model of the GRB emission is the optically thin synchrotron shock model (e.g., Piran 1999). A similar discussion in the following will be applied to the inverse Compton model. Let us consider a slow (rapid) shell with a Lorentz factor γ_s (γ_r), a mass m_s (m_r) and a width l_s (l_r). When a separation between two shells is L , the collision takes place at a radius $R_s \simeq 2L\gamma_s^2$. At the collision, the forward and the reverse shock are formed. Here we consider the reverse shock propagating into the rapid shell. The discussion for the forward shock is similar. We assume that a fraction of electrons ζ_e is accelerated in the shock to a power law distribution of Lorentz factor γ_e , $N(\gamma_e)d\gamma_e \propto \gamma_e^{-p}d\gamma_e$ for $\gamma_e \geq \gamma_{min} \equiv [(p-2)/(p-1)](\epsilon_e u'/\zeta_e n' m_e c^2)$, where n' and u' are the number density and the internal energy density in the local frame, respectively, $p \gtrsim 2$, and we assume that a

fraction ϵ_e of the internal energy goes into the electrons. We also assume that a fraction ϵ_B of the internal energy goes into the magnetic field, $B^2 = 8\pi\epsilon_B u'$. The local frame quantities, u' and n' , can be calculated using the shock jump conditions (Blandford & Mckee 1976; Sari & Piran 1995). We assume that the unshocked shells are cold and the shocked shells are extremely hot. If the Lorentz factor of the shocked region is γ , the relative Lorentz factor of the unshocked and the shocked region is given by $\gamma_{rel} \simeq (\gamma_r/\gamma + \gamma/\gamma_r)/2 \simeq \gamma_r/2\gamma$, so that $u' = (\gamma_{rel} - 1)n'm_p c^2 \simeq \gamma_{rel} n' m_p c^2$. The number densities of the unshocked and the shocked region are given by $n'_r = m_r/4\pi m_p R_s^2 l_r \gamma_r \simeq m_r/16\pi m_p L^2 \gamma_s^2 l_r \gamma_r$ and $n' = (4\gamma_{rel} + 3)n'_r \simeq 4\gamma_{rel} n'_r$, respectively. Thus, the characteristic synchrotron energy is given by

$$E_b = \frac{\hbar q_e B \gamma \gamma_{min}^2}{m_e c (1+z)} \simeq 260 \left[\frac{p-2}{p-1} \right]^2 \epsilon_e^2 \epsilon_B^{1/2} \zeta_e^{-2} m_{r,28}^{1/2} L_{10}^{-1} l_{r,9}^{-1/2} \gamma_{s,2}^{-2} \gamma_{r,2}^{1/2} \gamma_{rel}^2 (1+z)^{-1} \text{ keV}, \quad (3)$$

where we assume that the source is at a redshift z . Note that the relative Lorentz factor of the unshocked and shocked region γ_{rel} depends on the relative Lorentz factor of the rapid and slow shell $\gamma_{rs} \simeq (\gamma_r/\gamma_s + \gamma_s/\gamma_r)/2 \simeq \gamma_r/2\gamma_s$ and the ratio between the number densities in these shells $f \equiv n'_s/n'_r = m_s l_r \gamma_r / m_r l_s \gamma_s$ (Sari & Piran 1995). For the ultrarelativistic shock case $\gamma_{rs}^2 \gg f$, $\gamma_{rel} = \gamma_{rs}^{1/2} f^{1/4} / \sqrt{2} = (m_s l_r \gamma_r^3 / m_r l_s \gamma_s^3)^{1/4} / 2$.

Equation (3) shows that the break energy is written in the form of a product of many variables. For such a variable made from the product of many variables, the lognormal distribution may have a very simple origin, that is, the central limit theorem (Crow & Shimizu 1988; Montroll & Shlesinger 1982). Let a variable q be written in the form of a product of variables,

$$q = x_1 x_2 \cdots x_n. \quad (4)$$

Then,

$$\log q = \log x_1 + \log x_2 + \cdots + \log x_n. \quad (5)$$

When the individual distributions of $\log x_i$ satisfy certain weak conditions that include the existence of second moments, the central limit theorem is applicable to the variable $\log q$, so that the distribution function of $\log q$ tends to the normal distribution as n tends to infinity.

As an example, we numerically generated random variables x_i ($i = 1, 2, \dots$) whose logarithms are uniformly distributed between 0 and 1. Figure 3 shows the histogram of the product of these three variables, $q = x_1 x_2 x_3$, for 10^4 experimental realizations. The distribution of q agrees with the lognormal distribution quite well. It is surprising that the χ^2 test gives the probability of 0.483 (the reduced χ^2 is 1.00 with 278 degrees of freedom) that the distribution of q is taken from the lognormal distribution. This example shows

that the lognormal distributions may be achieved by a relatively small number of variables (Yonetoku & Murakami 2001). Note that when the number of the variables is two, i.e., $q = x_1 x_2$, the probability that the distribution is taken from the lognormal distribution was only 1.64×10^{-5} , so that a product of only one more variable may make a distribution lognormal.

Therefore the lognormal distribution of the break energy may be a natural result from the central limit theorem. We may say, “Astrophysically, not $n = \infty$ but $n = 3$ gives the lognormal distribution !!”. The effect of the redshift is just to add one variable in equation (3) if the redshift in the observed data is randomly chosen.

4. PULSE FLUENCE/DURATION/INTERVAL DISTRIBUTIONS

Let us consider the lognormal distributions in other quantities related to GRBs. When the rapid shell catches up the slow one in the internal shock, using the conservation of the energy and the momentum, the Lorentz factor of the merged shell γ_m and the internal energy E_{int} produced by the collision are given by $\gamma_m \simeq [(m_r \gamma_r + m_s \gamma_s)/(m_r/\gamma_r + m_s/\gamma_s)]^{1/2}$ and $E_{int} = m_r \gamma_r + m_s \gamma_s - (m_r + m_s) \gamma_m$, respectively (e.g., Piran 1999). If we assume that a fraction ϵ_e of the internal energy goes into the electrons and a fraction ϵ_w of the energy radiated by the electrons is within the gamma-ray band, the observed energy is given by

$$E_{obs} = \epsilon_w \epsilon_e E_{int} (1 + z)^{-1} \sim \epsilon_w \epsilon_e m_r \gamma_r (1 + z)^{-1}. \quad (6)$$

Equation (6) shows that the observed energy, which is proportional to the pulse fluence, is written in the form of a product of five variables,³ $\epsilon_w, \epsilon_e, m_r, [\gamma_r - \gamma_m + (m_s/m_r)(\gamma_s - \gamma_m)] \sim \gamma_r$ and $(1 + z)^{-1}$. Therefore the lognormal distribution of the pulse fluence may be a natural result from the central limit theorem.

The pulse duration is determined by three time scales: the hydrodynamic time scale, the cooling time scale, and the angular spreading time scale (Kobayashi, Piran & Sari 1997; Katz 1997; Fenimore, Madras & Sergei 1996). The cooling time scale is usually much shorter than the other two time scales in the internal shocks (Sari, Narayan & Piran 1996). The hydrodynamic time scale $\sim l/c$ and the angular spreading time scale determine the rise and the decay time of the pulse, respectively. Since most observed pulses rise more quickly than they decay (Norris et al. 1996), we assume that the pulse duration is mainly determined by

³Although the fraction ϵ_e of the internal energy that goes into electrons may be a fundamental constant, there is still a large dispersion of order unity in ϵ_e deduced from the the afterglow observations (Panaiteescu & Kumar 2001).

the angular spreading time $\simeq R_s/2c\gamma_m^2$. Then, the pulse duration δt is given by

$$\delta t \simeq (L/c)(\gamma_s^2/\gamma_m^2)(1+z). \quad (7)$$

On the other hand, the interval between pulses Δt is determined by the separation between shells,

$$\Delta t \simeq (L/c)(1+z), \quad (8)$$

since all shells are moving towards us with almost the speed of light (Kobayashi, Piran & Sari 1997; Nakar & Piran 2001).

Equation (8) shows that the pulse interval Δt reflects the separation between shells L , while equation (7) shows that the pulse duration δt is multiplied by one more factor $(\gamma_s/\gamma_m)^2$ other than $(L/c)(1+z)$. Therefore, if we consider that the distribution of a product of variables tends to the lognormal distribution as the number of the multiplied variables increases, the distribution of the pulse duration δt may be closer to the lognormal distribution than that of the pulse interval Δt . In fact, Nakar & Piran (2001) argued that the pulse duration δt has the lognormal distribution while the pulse interval Δt does not, as noticed by Li & Fenimore (1996). The pulse interval has an excess of long intervals relative to the lognormal distribution. This may suggest the existence of a different distribution, i.e., quiescent times (long periods with no activity) (Nakar & Piran 2001; Ramirez-Ruiz & Merloni 2001).⁴ But the central limit theorem may be also responsible for the lognormal distribution of the pulse duration δt .

5. DISCUSSIONS

We considered the possible origin of the lognormal distributions in the break energy, the pulse fluence and the pulse duration as a result of the central limit theorem. Astrophysically the lognormal distribution may be achieved by a product of only a few variables. The effect of the redshift is just to add one variable to the product so that the redshift distribution is hidden.

We have no idea about the origin of the lognormal distributions in the pulse interval⁵ Δt

⁴The correlation between the pulse interval δt and the pulse duration Δt is broken when the separation between shells is too large to collide each other before the external shock. However, almost all quiescent times in Nakar & Piran (2001) are smaller than the limit $\delta t \lesssim 100 \text{ sec } E_{52}^{1/3} n_1^{-1/3} \gamma_{100}^{-8/3}$ so that shells may collide.

⁵Note that the distribution of the pulse interval is lognormal if we exclude the quiescent times (Nakar & Piran 2001).

and the total duration ΔT . However the viewing angle may be one factor to be multiplied to the pulse interval Δt . Recently we suggested that the luminosity-lag relation could be explained by the variation in the viewing angle θ_v from the axis of the jet (Ioka & Nakamura 2001; Nakamura 2000). The duration of the pulse from the jet also depends on the viewing angle, and according to Figure 2 of Ioka & Nakamura (2001) we have

$$\Delta t \propto (L/c)(1+z)(1+\gamma^2\theta_v^2), \quad (9)$$

when $\theta_v \sim \Delta\theta$ where γ is the Lorentz factor of the jet and $\Delta\theta$ is the opening half-angle of the jet. The multiplied factor $(1+\gamma^2\theta_v^2)$ may be responsible for the lognormal distribution of the pulse interval Δt . Note that the total duration ΔT is equal to the lifetime of the central engine and thus does not depend on θ_v .

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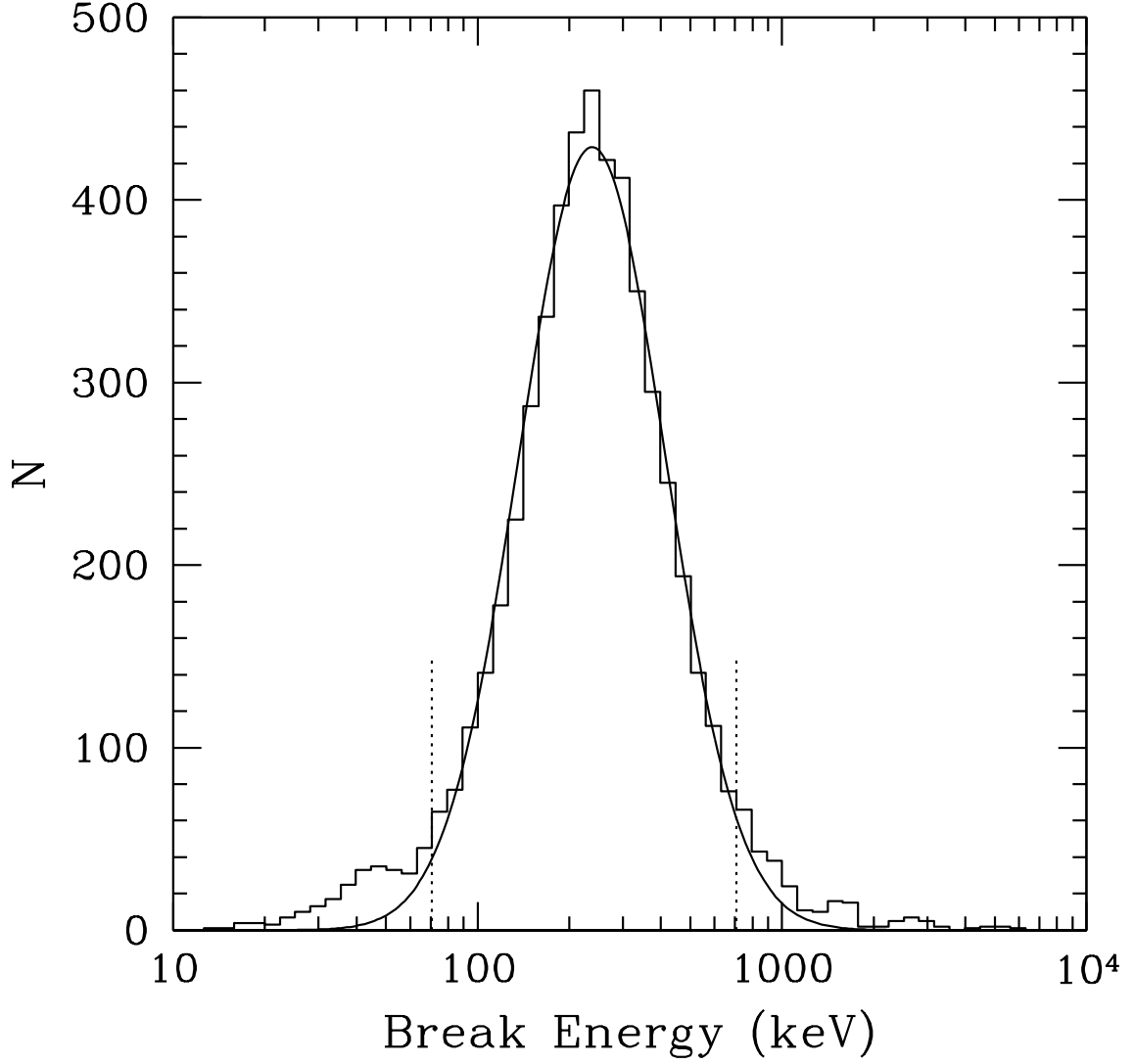


Fig. 1.— The break energy distribution taken from the electronic edition of Preece et al. (2000) is shown. The lognormal fitting of the data between 70.8 keV and 708 keV is shown by a solid line. The fitting range between 70.8 keV and 708 keV is shown by dotted lines. The χ^2 test gives the probability of 0.497 (the reduced χ^2 is 0.963 with 17 degrees of freedom) that the data was taken from the lognormal distribution, with the mean $\mu = 2.38 \pm 0.004$ ($E_b \simeq 238$ keV) and the standard deviation $\sigma = 0.240 \pm 0.004$ (1σ width is between 137 keV and 413 keV).

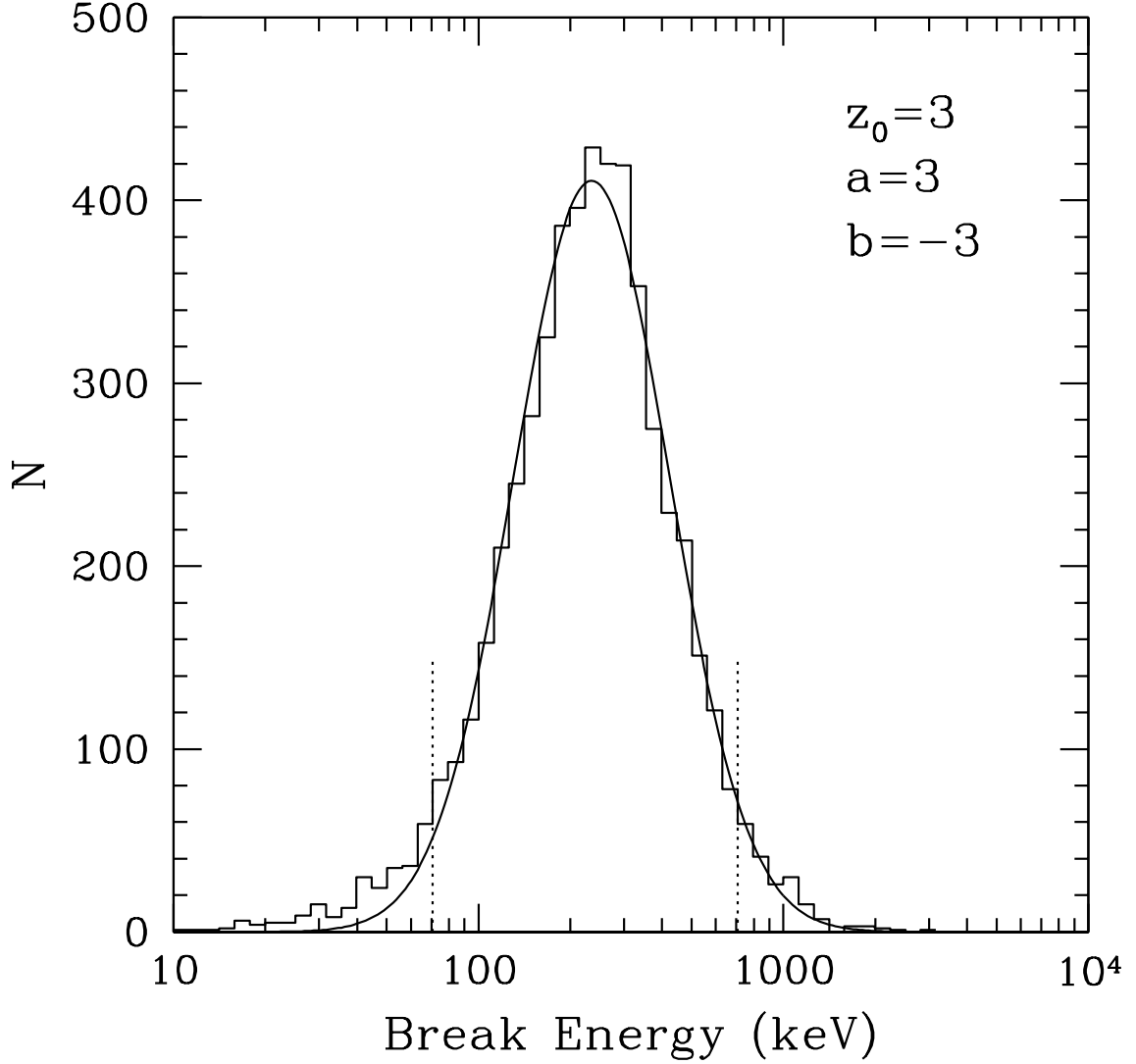


Fig. 2.— One experimental realization of the observed break energy distribution for $(z_0, a, b) = (3, 3, -3)$ in equation (2) is shown. The lognormal fitting of the data between 70.8 keV and 708 keV is shown by a solid line. The fitting range between 70.8 keV and 708 keV is shown by dotted lines. The χ^2 test gives the probability of 0.128 (the reduced χ^2 is 1.39 with 17 degrees of freedom) that the data was taken from the lognormal distribution, with $\mu = 2.37 \pm 0.004$ ($E_b \simeq 235$ keV) and $\sigma = 0.256 \pm 0.004$ (1σ width is between 130 keV and 423 keV). There is an excess of soft bursts relative to the lognormal fit as in Figure 1.

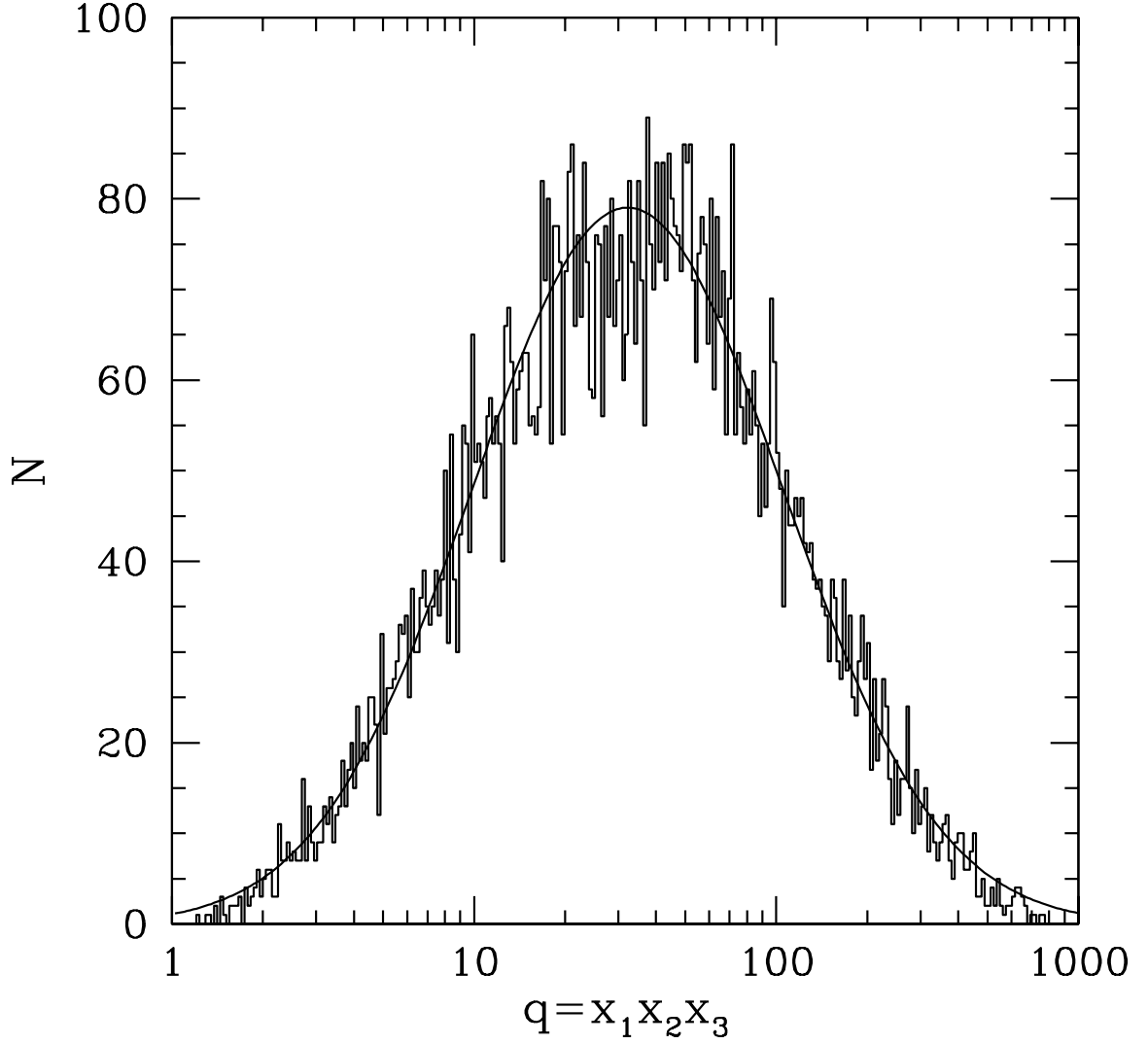


Fig. 3.— The distribution of the product of three random variables $q = x_1 x_2 x_3$, each distributing uniformly between 0 and 1 in the logarithmic space (that is $0 < \log x_i < 1$), for 10^4 experimental realizations is shown. The lognormal fitting of the data is shown by a solid line. The χ^2 test gives the probability of 0.483 (the reduced χ^2 is 1.00 with 278 degrees of freedom) that the distribution of q is taken from the lognormal distribution.